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# THE EFFECT OF SURFACE CURVATURE AND DISCONTINUITY ON THE SURFACE ENERGY DENSITY AND OTHER INDUCED FIELDS IN ELASTIC DIELECTRICS WITH POLARIZATION GRADIENT

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**Abstract**—Mindlin's theory of elastic dielectrics with polarization gradient is used along with the values of material coefficients obtained by Askar *et al.* to solve two types of problems involving cylindrical and spherical cavities and a free linear crack in "plane-strain". In the case of the two cavities the surface energy density of deformation and polarization is found to be changed by an amount directly proportional to a length parameter $l_1$, of order of magnitude of the interatomic distance, and inversely proportional to the radius of curvature of the cavity. As for the crack a stress singularity of the order $\varepsilon^{-\frac{1}{2}}$ as $\varepsilon \to 0$ is obtained in the absence of any external forces. This singularity is of the same order as those given by classical elasticity and couple stress theories. However, the surface energy density of the crack surface is bounded.

## INTRODUCTION

IN MINDLIN'S theory of elastic dielectrics [1] the internal energy density of deformation and polarization is a function of strain, polarization and polarization gradient. This theory yields a set of linear field equations in which the displacement, polarization and the electric potential of the Maxwell self-field are all coupled through the constitutive relations even for the cases of centrosymmetric and isotropic materials. Furthermore, due to the term linear in polarization gradient in the energy density function this theory can account for a surface energy density of deformation and polarization, which depends on the field variables. In another paper [2], Mindlin shows, by means of one-dimensional example, that his equations, rather than the classical equations of elastic dielectrics, are the long wave approximation to the equations of a lattice of polarizable atoms. He also exhibited the relations between the force constants of the one-dimensional lattice and the pertinent coefficients in the constitutive equations of his continuum theory. Askar *et al.* [3] have established the relationship between this continuum formulation and the lattice formulation using Dick and Overhauser's [4] shell model and have obtained the numerical values of the material coefficients for a number of alkali halides, by means of the long wave approximation of the discrete analysis.

Since the surface effects are directly related to the field variables, which in turn, depend on the boundary conditions of a specific problem, this continuum theory permits the study of effects of surface curvature and discontinuity on the surface energy density of deformation and polarization and other induced fields.

In this article, first problems with cylindrical and spherical cavities are solved and the influence of the surface geometry is deduced from a comparison of the curved cases

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with the plane case solved by Mindlin [1]. Secondly, the free linear crack in an infinite medium is considered and the investigation of the fields along the crack axis yields singularities of the type $\varepsilon^{\frac{1}{2}}$ as $\varepsilon \to 0$ at the crack tip. However the surface energy density remains bounded on the crack surface. The mathematical treatment consists of uncoupling the field equations after introducing the Helmholtz decomposition of the displacement and polarization fields. The crack problem is attacked by reducing it to two pairs of simultaneous dual integral equations which, further, are reduced to a system of algebraic equations by a procedure due to Erdogan and Bahar [5]. A similar problem involving more than one field variable has been treated by Sternberg and Muki [6] in studying the effect of the couple stresses on fields around a crack.

## 1. POLARIZATION GRADIENT AND SURFACE ENERGY IN ELASTIC DIELECTRICS

For an elastic dielectric in vacuum occupying a volume $V$ and bounded by a surface $S$, the governing equilibrium equations, and the boundary conditions, in the absence of an external body force and an external electric field, are [1]:

$$t_{ij,i} = 0, \qquad E_{ij,i} + \bar{E}_j - \phi_{,j} = 0, \qquad -\varepsilon_0 \phi_{,ii} + P_{i,i} = 0, \qquad \text{in } V,$$
$$\varepsilon_0 \phi_{,ii} = 0, \qquad \text{in vacuum}, \tag{1.1}$$

and

$$n_i t_{ij} = t_j, \qquad n_i E_{ij} = 0, \qquad n_i[\varepsilon_0(\phi^+_{,i} - \phi^-_{,i}) + P_i] = 0, \qquad \text{on } S, \tag{1.2}$$

where $t_{ij}$, $\bar{E}_i$ are the stress and effective local electric force, $E_{ij}$ is derivable from the energy density of deformation and polarization $W^L(E_{ij} \equiv \partial W^L/\partial P_{j,i})$, $\phi$ is the potential of Maxwell self-field, $P_i$ is the polarization, $\varepsilon_0$ is the permittivity of vacuum, $n_i$ is the unit outward normal to $S$, $\phi^+$, $\phi^-$ are the limits of $\phi$ from the positive and negative sides of $S$. The constitutive relations, for isotropic media are [1]

$$t_{ij} = c_{12} u_{k,k} \delta_{ij} + c_{44}(u_{i,j} + u_{j,i}) + d_{12} P_{k,k} \delta_{ij} + d_{44}(P_{j,i} + P_{i,j}),$$
$$E_{ij} = d_{12} u_{k,k} \delta_{ij} + d_{44}(u_{i,j} + u_{j,i}) + b_{12} P_{k,k} \delta_{ij} + b_{44}(P_{j,i} + P_{i,j}) + b_{77}(P_{j,i} - P_{i,j}) + b^0 \delta_{ij}, \tag{1.3}$$
$$\bar{E}_i = -a P_i,$$

where $u_i$ is the displacement and $c_{12}, c_{44}, d_{12}, d_{44}, b_{12}, b_{44}, b_{77}, a$ and $b^0$ are the material constants.

The surface energy density $T$ is

$$T = \tfrac{1}{2} b^0 [P_i n_i]_S. \tag{1.4}$$

Substitution of the relations (1.3) into the equilibrium equations (1.1) results in the following coupled system of equations, in vector notations, in seven variables $\mathbf{u}$, $\mathbf{P}$ and $\phi$:

$$c_{44}\nabla^2\mathbf{u} + (c_{12} + c_{44})\nabla\nabla \cdot \mathbf{u} + d_{44}\nabla^2\mathbf{P} + (d_{12} + d_{44})\nabla\nabla \cdot \mathbf{P} = 0,$$
$$d_{44}\nabla^2\mathbf{u} + (d_{12} + d_{44})\nabla\nabla \cdot \mathbf{u} + (b_{44} + b_{77})\nabla^2\mathbf{P} + (b_{12} + b_{44} - b_{77})\nabla\nabla \cdot \mathbf{P} - a\mathbf{P} - \nabla\phi = 0,$$
$$-\varepsilon_0 \nabla^2\phi + \nabla \cdot \mathbf{P} = 0, \qquad \text{in } V, \tag{1.5}$$
$$\nabla^2\phi = 0, \qquad \text{in vacuum}.$$

By introducing the Helmholtz decomposition [7, 8]

$$\mathbf{u} = \nabla\psi + \nabla\times\mathbf{H}, \qquad \nabla.\mathbf{H} = 0, \qquad \mathbf{P} = \nabla\chi + \nabla\times\mathbf{K}, \qquad \nabla.\mathbf{K} = 0, \tag{1.6}$$

into the "displacement" equations (1.5), and after some eliminations, the following system of uncoupled equations is obtained.
In $V$:

$$\begin{aligned} &(\nabla^2 - l_1^{-2})\nabla^4\psi = 0, \quad (\nabla^2 - l_2^{-2})\nabla^4\mathbf{H} = 0, \quad \nabla.\mathbf{H} = 0, \\ &(\nabla^2 - l_1^{-2})\nabla^2\chi = 0, \quad (\nabla^2 - l_2^{-2})\nabla^2\mathbf{K} = 0, \quad \nabla.\mathbf{K} = 0, \quad (\nabla^2 - l_1^{-2})\nabla^2\phi = 0, \end{aligned} \tag{1.7}$$

and in vacuum:

$$\nabla^2\phi = 0,$$

where

$$l_1^2 = \frac{b_{11}(d_{11}^2/c_{11})}{a+\varepsilon_0^{-1}}, \qquad l_2^2 = \frac{(b_{44}+b_{77})-(d_{44}^2/c_{44})}{a}$$
$$c_{11} = c_{12}+2c_{44}, \qquad d_{11} = d_{12}+2d_{44}, \qquad b_{11} = b_{12}+2b_{44}. \tag{1.8}$$

The quantities $l_1^2$, $l_2^2$ defined in equations (1.8) must be positive due to the requirement of positive definiteness of $W^L$. Their numerical values computed from the material coefficients, which are obtained by the application of the Reuss method of averaging [9] to the material constants derived from lattice model [3], are positive as shown in Table 1 [10].

During these operations the order of the differential equations is artificially increased. In order to have a determinate problem, further restrictions, in addition to the boundary conditions, are brought in by requiring the compatibility of the solution with the original

TABLE 1. ISOTROPIC MATERIAL COEFFICIENTS AND RELATED PARAMETERS

| | Material | NaCl | KCl |
|---|---|---|---|
| $c_{12}$ | $\frac{10^{12}\,\text{dyn}}{\text{cm}^2}$ | 0·148 | 0·105 |
| $c_{44}$ | | 0·149 | 0·105 |
| $d_{12}$ | $\frac{10^7\,\text{dyn cm}}{\text{C}}$ | 0·470 | 0·392 |
| $d_{44}$ | | −0·170 | −0·178 |
| $b_{12}$ | $\frac{10^4\,\text{dyn cm}^4}{\text{C}^2}$ | $-1\cdot6\times10^{-7}$ | $-25\cdot6\times10^{-7}$ |
| $b_{44}$ | | 0·344 | 0·600 |
| $b_{77}$ | | 0·344 | 0·600 |
| $-b^0$ | $\frac{10^7\,\text{dyn cm}}{\text{C}}$ | 1·44 | 1·29 |
| $a$ | $\frac{10^9\,\text{dyn cm}^2}{\text{C}^2}$ | 1·74 | 2·43 |
| $l_1^2$ | $10^{-16}\,\text{cm}^2$ | 0·527 | 0·873 |
| $l_2^2$ | | 3·943 | 4·926 |
| $\alpha$ | | 0·995 | 0·995 |
| $\beta$ | | 2·299 | 5·944 |

equations (1.5). Also, due to the nature of the potentials, they can only be determined up to an arbitrary constant which may be set equal to zero.

## 2. FREE SURFACE PROBLEMS

### 2.1 *Half space problem*

This problem has been solved by Mindlin [1] for the case of centrosymmetric cubic crystals. The solution for the isotropic case is analogous to that of Mindlin's. However it is presented here as a reference for the comparison of the curved surface cases and to make some estimates of the effects due to the surface geometry.

Consider the dielectric occupying the region $x_1 \geq 0$ with $x_2, x_3$ axes lying on the free surface. At $x_1 = 0$ the free surface conditions, according to equations (1.2), are

$$t_{11} = t_{12} = t_{13} = 0, \qquad E_{11} = E_{12} = E_{13} = 0, \qquad \varepsilon_0(\phi^+_{,1} - \phi^-_{,1}) + P_1 = 0. \tag{2.1}$$

In addition, the fields are required to be regular at infinity.

Due to the symmetrical nature of the problem, one may assume

$$\psi = \psi(x_1), \qquad \chi = \chi(x_1), \qquad \phi = \phi(x_1), \qquad \mathbf{H} = 0, \qquad \mathbf{K} = 0. \tag{2.2}$$

Using (2.2) in the uncoupled equations (1.7) and requiring compatibility with the original equations (1.5), one has:

$$\begin{aligned} u_1 &= -\frac{b^0 d_{11}}{l_1(a+\varepsilon_0^{-1})c_{11}} e^{-x_1/l_1}, \\ P_1 &= \frac{b^0}{l_1(a+\varepsilon_0^{-1})} e^{-x_1/l_1}, \\ \phi &= -\frac{\varepsilon_0^{-1} b^0}{(a+\varepsilon_0^{-1})} e^{-x_1/l_1}, \qquad \text{for } x_1 \geq 0, \\ \phi &= 0, \qquad \text{for } x_1 < 0. \end{aligned} \tag{2.3}$$

The surface energy density and the nonvanishing stress components are

$$\begin{aligned} T &= -\frac{(b^0)^2}{2l_1(a+\varepsilon_0^{-1})}, \\ t_{22} &= t_{33} = c_{11}\frac{b^0}{d_{11}}\left(\frac{c_{12}}{c_{11}} - \frac{d_{12}}{d_{11}}\right)\left(\frac{b_{11}c_{11}}{d_{11}d_{11}} - 1\right)^{-1} e^{-x_1/l_1}. \end{aligned} \tag{2.4}$$

It can be seen that $t_{22}$ is of the same order of magnitude as $c_{11}$ (which is $\lambda + 2\mu$ in terms of the Lame constants), the elastic stiffness of the material, since the product of dimensionless ratios of material coefficients in the last equation of (2.4) is of the same order of magnitude according to the values given in Table 1.

The stress decays very fast away from the boundary since the parameter $l_1$, as calculated in [3], is of the order of magnitude of the interatomic distance as estimated from electron diffraction data [11].

For convenience of comparison in the following sections, the quantities in (2.3) and (2.4) on the surface are defined as

$$\begin{aligned} &u^0 \equiv u_1(x_1 = 0), \qquad P^0 \equiv P_1(x_1 = 0), \qquad \phi^0 \equiv \phi(x_1 = 0), \\ &T^0 \equiv T, \qquad t_{22}^0 = t_{33}^0 \equiv t_{22}(x_1 = 0). \end{aligned} \tag{2.5}$$

2.2 *Cylindrical cavity problem*

Consider an infinite medium containing a cylindrical cavity whose axis coincides with the $z$ axis. In the plane perpendicular to $z$, the polar coordinates $r, \theta$ are used. At $r = R$, the surface of the cylindrical cavity, the free surface conditions are:

$$t_{rr} = t_{r\theta} = t_{rz} = 0, \qquad E_{rr} = E_{r\theta} = E_{rz} = 0, \qquad \varepsilon_0\left(\frac{\partial\phi^+}{\partial r} - \frac{\partial\phi^-}{\partial r}\right) + P_r = 0. \tag{2.6}$$

In addition, the fields are required to be regular at infinity, $r \to \infty$, and $\varphi$ to be regular on the axis of the cylinder, $r = 0$.

Because of the symmetry, one may assume

$$\begin{aligned} &\psi = \psi(r), \qquad \chi = \chi(r), \qquad \phi = \phi(r), \qquad \mathbf{H} = 0, \qquad \mathbf{K} = 0, \\ &\nabla^2 = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{dr}. \end{aligned} \tag{2.7}$$

As in Section 2.1, one has:

$$\begin{aligned} u_r &= u^0\left\{\frac{K_1(r/l_1) + \beta(R/r)K_1(R/l_1)}{[K_0(R/l_1) + \alpha(l_1/R)K_1(R/l_1)]}\right\}, \\ P_r &= P^0\left\{\frac{K_1(r/l_1)}{[K_0(R/l_1) + \alpha(l_1/R)K_1(R/l_1)]}\right\}, \\ \phi &= \phi^0\left\{\frac{K_0(r/l_1)}{[K_0(R/l_1) + \alpha(l_1/R)K_1(R/l_1)]}\right\}, \qquad r \geq R, \\ \phi &= 0, \qquad r < R. \end{aligned} \tag{2.8}$$

where $u^0$, $P^0$, $\phi^0$ correspond to the fields at the free flat surface as defined in equation (2.5), $K_0(x)$, $K_1(x)$ are modified Bessel functions of the second kind and

$$\begin{aligned} \alpha &= 2\left(b_{44} - \frac{d_{44}^2}{c_{44}}\right) \bigg/ \left(b_{11} - \frac{d_{11}^2}{c_{11}}\right), \\ \beta &= 1 - \frac{c_{11}}{c_{44}}\frac{d_{44}}{d_{11}}. \end{aligned} \tag{2.9}$$

The surface energy density and the nonvanishing stress components are

$$T = T^0 \frac{K_1(R/l_1)}{K_0(R/l_1)+\alpha(l_1/R)K_1(R/l_1)},$$
$$t_{\theta\theta} = t^0_{22}\frac{K_0(r/l_1)+(l_1/r)K_1(r/l_1)-(R/r)(l_1/r)K_1(r/l_1)}{K_0(R/l_1)+\alpha(l_1/R)K_1(R/l_1)}, \tag{2.10}$$
$$t_{zz} = t^0_{33}\frac{K_0(r/l_1)}{K_0(R/l_1)+\alpha(l_1/R)K_1(R/l_1)},$$

where $T^0$, $t^0_{22}$, $t^0_{33}$ correspond to the surface energy, stress components at the flat free surface as defined in equation (2.5).

As mentioned in the previous section, $l_1$ is of the order of magnitude of the interatomic distances so that, in the domain of validity of the continuum hypothesis, $R/l_1$, $r/l_1$ are large numbers. The use of the asymptotic representation $(\pi/2x)^{\frac{1}{2}}\,e^{-x}$ for the Bessel function $K_n(x)$ results in

$$T = T^0\left(1-\alpha\frac{l_1}{R}\right). \tag{2.11}$$

Thus the absolute value of surface energy density is reduced by the curvature of an interior cylindrical surface as compared to the case of a flat surface since $\alpha$ is positive for the materials shown in Table 1.

As the pressure is proportional to the negative of the spherical part of the stress tensor

$$p = -\tfrac{1}{3}t_{ii}, \tag{2.12}$$

the "pressure" at the surface of a cylindrical cavity can be written as

$$p = p^0 - \frac{4}{3}\left(\frac{d_{12}}{d_{11}}-\frac{c_{12}}{c_{11}}\right)\frac{d_{11}}{b^0}\alpha\frac{T^0}{R}, \tag{2.13}$$

where $p^0$ is the pressure at a flat surface, and can be obtained by means of equations (2.12) and (2.4). The form of this result is similar to that of the classical Laplace formula [12]. Since the second term in equation (2.13) is positive according to the values given in Table 1, the pressure at a cylindrical interior surface is reduced by an amount which is directly proportional to the surface energy density $T^0$ and inversely proportional to the radius of curvature.

### 2.3 *Spherical cavity*†

Consider an infinite medium containing a spherical cavity whose center is chosen as the origin of the spherical coordinate system $(r, \theta, \omega)$. At the surface of the cavity, $r = R$, the free surface conditions are:

$$t_{rr} = t_{r\theta} = t_{r\omega} = 0, \qquad E_{rr} = E_{r\theta} = E_{r\omega} = 0, \qquad \varepsilon^0\left(\frac{\partial\phi^+}{\partial r}-\frac{\partial\phi^-}{\partial r}\right)+P_r = 0. \tag{2.14}$$

† It has come to our attention, after the preparation of the manuscript, that this problem has been treated by Schwartz [13]. The two results coincide by noticing that for the modified Bessel function of order $\frac{1}{2}$, $x^{-\frac{1}{2}}K_{\frac{1}{2}}(x) = \sqrt{(\pi/2)}x^{-1}e^{-x}$.

In addition, the fields are required to be regular at infinity, $r \to \infty$, and $\varphi$ to be regular at the center of the sphere, $r = 0$.

In a similar manner as in the preceding sections, one obtains

$$\begin{aligned} u_r &= u^0 \frac{(l_1/r)^{\frac{1}{2}} K_{3/2}(r/l_1) + \beta (R/r)^2 (l_1/R)^{\frac{1}{2}} K_{3/2}(R/l_1)}{[(l_1/R)^{\frac{1}{2}} K_{1/2}(R/l_1) + 2(l_1/R)^{\frac{1}{2}} K_{3/2}(R/l_1)]}, \\ P_r &= P^0 \frac{(l_1/r)^{\frac{1}{2}} K_{3/2}(r/l_1)}{[(l_1/R)^{\frac{1}{2}} K_{1/2}(R/l_1) + 2\alpha (l_1/R)^{3/2} K_{3/2}(R/l_1)]}, \\ \phi &= \phi^0 \frac{(l_1/r) K_{1/2}(r/l_1)}{[(l_1/R)^{\frac{1}{2}} K_1(R/l_1) + 2\alpha (l_1/R)^{3/2} K_{3/2}(R/l_1)]}, \end{aligned} \tag{2.15}$$

where $K_{1/2}(x)$, $K_{3/2}(x)$ are modified Bessel functions of the second kind and $\alpha, \beta$ are defined in equation (2.9).

The surface energy density and the "pressure" at the boundary for large values of $(R/l_1)$, by the similar operations to those in the previous section, are:

$$\begin{aligned} T &= T^0 \left(1 - \alpha \frac{2l_1}{R}\right), \\ p &= p^0 - \frac{4}{3}\left(\frac{d_{12}}{d_{14}} - \frac{c_{12}}{c_{14}}\right) \frac{d_{14}}{b^0} \alpha \frac{2T^0}{R}. \end{aligned} \tag{2.16}$$

## 3. CRACK PROBLEM

### 3.1 *Formulation as a mixed boundary value problem*

Consider an infinite elastic dielectric, containing a free linear crack at the surface defined by

$$x = 0, \qquad -L < y < L, \qquad -\infty < z < \infty. \tag{3.1}$$

The medium is free of all external effects and the surface energy is the only source which induces mechanical and electrical fields. The "plane strain" problem to be considered, is defined as:

$$u_z = 0, \qquad P_z = 0, \tag{3.2}$$

and all other non-zero field quantities depend on $x$, $y$ only.

This boundary value problem is governed by equations (1.5) and subject to the free boundary conditions, for $0 \leq |y| < L$

$$\begin{aligned} &t_{xx}(0, y) = t_{xy}(0, y) = 0, \qquad E_{xx}(0, y) = E_{xy}(0, y) = 0, \\ &\frac{\partial \phi^+(0, y)}{\partial x} - \frac{\partial \phi^-(0, y)}{\partial x} + \varepsilon_0^{-1} P_x(0, y) = 0. \end{aligned} \tag{3.3}$$

In addition, the fields are required to be regular at infinity and $\phi^+$ is required to be regular inside the crack.

In view of the symmetry of the problem about the $y$ axis, as for crack problems in classical elasticity [14] and couple–stress [6] theories, one may assume that $u_x$, $P_x$ and $\partial\phi/\partial x$ are functions odd in $x$ and even in $y$ while $u_y$, $P_y$ and $\partial\phi/\partial y$ are functions even in $x$

and odd in $y$. Thus the solution of the mixed boundary value problem for the half space $x \geq 0$, subject to following conditions at $x = 0$

$$
\begin{aligned}
&t_{xy}(0, y) = 0, \qquad E_{xy}(0, y) = 0, \\
&\frac{\partial \phi^{+}(0, y)}{\partial x} - \frac{\partial \phi^{-}(0, y)}{\partial x} + \varepsilon_0^{-1} P_1(0, y) = 0, \qquad 0 \leq |y| < \infty, \\
&t_{xx}(0, y) = 0, \qquad E_{xx}(0, y) = 0, \qquad 0 \leq |y| < L, \\
&u_x(0, y) = 0, \qquad P_x(0, y) = 0, \qquad L \leq |y|,
\end{aligned}
\tag{3.4}
$$

together with the regularity conditions.

### 3.2 *Auxiliary problem*

In preparation for the solution of the mixed boundary value problem defined by (3.4), consider first, the boundary value problem specified by, for $0 \leq |y| < \infty$,

$$
\begin{aligned}
&t_{xy}(0, y) = 0, \qquad E_{xy}(0, y) = 0, \\
&\frac{\partial \phi^{+}(0, y)}{\partial x} - \frac{\partial \phi^{-}(0, y)}{\partial x} + \varepsilon_0^{-1} P_x(0, y) = 0, \\
&u_x(0, y) = f(y), \qquad P_x(0, y) = g(y),
\end{aligned}
\tag{3.5}
$$

together with the regularity conditions at infinity. The solution of this problem in terms of the unspecified functions $f$ and $g$, which are even functions of $y$, coincides with the solution of the free crack problem provided that $f$, $g$ satisfy the boundary conditions of the equivalent mixed boundary value problem, equations (3.4).

The foregoing auxiliary problem can be solved by choosing the potentials as

$$
\begin{aligned}
&\psi = \psi(x, y), \qquad \chi = \chi(x, y), \\
&\mathbf{H} = H(x, y)\mathbf{e}_z, \qquad \mathbf{K} = K(x, y)\mathbf{e}_z,
\end{aligned}
\tag{3.6}
$$

and by the use of Fourier integral transform and its inverse defined as

$$
\begin{aligned}
\bar{g}(x, \eta) &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} g(x, y)\, e^{-i\eta y}\, dy, \\
g(x, y) &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \bar{g}(x, \eta)\, e^{i\eta y}\, d\eta.
\end{aligned}
\tag{3.7}
$$

By substituting equations (3.6) in the uncoupled equations (1.7), using the constitutive equations (1.3), the boundary conditions (3.5) with the regularity conditions at infinity, checking the compatibility conditions with the original field equations (1.5), and applying the Fourier transform with the considerations of the symmetry of the functions, one obtains, for $x \geq 0$

$$\psi(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ [e_1 - 1 + e_1 x\eta)\Omega_1 - e_4 l_2^2 \eta^2 (\Omega_1 - \Omega_2)] \frac{\bar{f}(\eta)}{\eta} + [(e_2 - 1 + e_2 x\eta + e_3 e_6 l_2^2 \eta^2)\Omega_1 \right.$$

$$\left. + \Omega_2 + (e_5 - e_6) l_2^2 \eta^2 (\Omega_1 - \Omega_2)] \frac{d_{11}}{c_{11}} \frac{\bar{g}(\eta)}{\eta} \right\} \cos \eta y \, d\eta,$$

$$H(x, y) = -\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ x\eta\Omega_1 \frac{\bar{f}(\eta)}{\eta} + (x\eta\Omega_1 + e_6 l_2^2 \eta^2 \Omega_5) \frac{d_{44}}{c_{44}} \frac{\bar{g}(\eta)}{\eta} \right\} \sin \eta y \, d\eta,$$

$$\chi(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ e_4 l_2^2 \eta^2 (\Omega_1 - \Omega_2) \frac{c_{11}}{d_{11}} \frac{\bar{g}(\eta)}{\eta} - [(1 + e_5 l_2^2 \eta^2)\Omega_2 \right.$$

$$\left. + e_5 l_2^2 \eta^2 (\Omega_1 - \Omega_2)] \frac{\bar{g}(\eta)}{\eta} \right\} \cos \eta y \, d\eta, \tag{3.8}$$

$$K(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty e_5 l_2^2 \eta^2 \Omega_5 \frac{\bar{g}(\eta)}{\eta} \sin \eta y \, d\eta,$$

$$\phi(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \varepsilon_0^{-1} \left\{ e_4 l_2^2 \eta^2 (\Omega_1 - \Omega_2) \frac{c_{11}}{d_{11}} \frac{\bar{f}(\eta)}{\eta} \right.$$

$$\left. - [\Omega_2 + (e_5 - e_6) l_2^2 \eta^2 (\Omega_1 - \Omega_2)] \frac{\bar{g}(\eta)}{\eta} \right\} \cos \eta y \, d\eta,$$

and for $x < 0$

$$\phi = 0,$$

where

$$e_1 = c_{44}/c_{11}, \qquad e_2 = d_{44}/d_{11}, \qquad e_3 = \frac{d_{44}}{d_{11}} \frac{c_{11}}{c_{44}},$$

$$e_4 = 2\left(\frac{d_{44}}{d_{11}} - \frac{c_{44}}{c_{11}}\right) \frac{d_{11}^2}{c_{11}(a + \varepsilon_0^{-1}) l_2^2},$$

$$e_5 = 2\left(\frac{d_{11}}{c_{11}} - \frac{d_{44}}{c_{44}}\right) \frac{d_{44}}{(a + \varepsilon_0^{-1}) l_2^2} + \frac{e_6}{(1 + a\varepsilon_0)},$$

$$e_6 = 2\left[1 + \frac{b_{77}}{b_{44}}\left(1 - \frac{d_{44}^2}{b_{44} c_{44}}\right)^{-1}\right]^{-1}, \tag{3.9}$$

$$\Omega_1(x, \eta) = e^{-\eta x}, \qquad \Omega_2(x, \eta) = \frac{\eta}{(\eta^2 + l_1^{-2})^{\frac{1}{2}}} e^{-(\eta^2 + l_1^{-2})^{\frac{1}{2}} x},$$

$$\Omega_3(x, \eta) = e^{-(\eta^2 + l_1^{-2})^{\frac{1}{2}} x}, \qquad \Omega_4(x, \eta) = \frac{\eta}{(\eta^2 + l_2^{-2})^{\frac{1}{2}}} e^{-(\eta^2 + l_2^{-2})^{\frac{1}{2}} x},$$

$$\Omega_5(x, \eta) = e^{-(\eta^2 + l_2^{-2})^{\frac{1}{2}} x}.$$

Using the definitions given by (1.6) and the constitutive relations (1.3), the solution of the auxiliary problem in terms of $\bar{f}(\eta)$ and $\bar{g}(\eta)$ is

$$u_x(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{[((1-e_1)x\eta+1)\Omega_1+e_4l_2^2\eta^2(\Omega_1-\Omega_3)]\bar{f}(\eta)+[(1-e_1)e_3x\eta\Omega_1 + (1-(e_5-e_6)l_2^2\eta^2)(\Omega_1-\Omega_3)-e_3e_6l_2^2\eta^2(\Omega_1-\Omega_5)]\frac{d_{11}}{c_{11}}\bar{g}(\eta)\Big\} \cos\eta y\, d\eta,$$

$$u_y(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{[(e_1-(1-e_1)x\eta)\Omega_1-e_4l_2^2\eta^2(\Omega_1-\Omega_2)]\bar{f}(\eta) + [e_3((1+e_1)-(1-e_1)x\eta)\Omega_1-(1-(e_5-e_6)l_2^2\eta^2)(\Omega_1-\Omega_2) + e_3l_2^2\eta^2(\Omega_1-\eta^{-2}(\eta^2+l_2^{-2})\Omega_4)]\frac{d_{11}}{c_{11}}\bar{g}(\eta)\Big\} \sin\eta y\, d\eta,$$

$$P_x(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{-e_4l_2^2\eta^2(\Omega_1-\Omega_3)\frac{c_{11}}{d_{11}}\bar{f}(\eta) + [\Omega_3+e_5l_2^2\eta^2(\Omega_1-\Omega_3)+e_6l_2^2\eta^2(\Omega_3-\Omega_5)]\bar{g}(\eta)\Big\} \cos\eta y\, d\eta,$$

$$P_y(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{e_4l_2^2\eta^2(\Omega_1-\Omega_2)\frac{c_{11}}{d_{11}}\bar{f}(\eta)-[\Omega_2+e_4l_2^2\eta^2(\Omega_1-\Omega_2) + e_6l_2^2\eta^2(\Omega_2-\eta^{-2}(\eta^2+l_2^{-2})\Omega_4)]\bar{g}(\eta)\Big\} \sin\eta y\, d\eta,$$

$$t_{xx}(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{[-A_1(x\eta+1)\Omega_1-A_2e_4l_2^2\eta^2(\Omega_1-\Omega_2)]c_{11}\eta\bar{f}(\eta) + [A_1(x\eta+1)\Omega_1+A_2\Omega_2+A_2(e_5-e_6)l_2^2\eta^2(\Omega_1-\Omega_2)]d_{11}\eta\bar{g}(\eta)\Big\} \cos\eta y\, d\eta,$$

$$t_{yy}(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{[A_1(x\eta-1)\Omega_1-A_2e_4l_1^2l_2^{-2}\Omega_2 + A_2e_4l_2^2\eta^2(\Omega_1-\Omega_2)]c_{11}\eta\bar{f}(\eta)+[(A_4+A_1x\eta)\Omega_1 + (A_3-A_2l_1^{-2}\eta^{-2})\Omega_2-A_2(e_5-e_6)l_2^2\eta^2(\Omega_1-\Omega_2)]d_{11}\eta\bar{g}(\eta)\Big\} \cos\eta y\, d\eta, \tag{3.10}$$

$$t_{xy}(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{[A_1x\eta\Omega_1+A_2e_4l_2^2\eta^2(\Omega_1-\Omega_3)]c_{11}\eta\bar{f}(\eta) + A_2(1-(e_5-e_6)l_2^2\eta^2)(\Omega_1-\Omega_3)]d_{11}\eta\bar{g}(\eta)]\Big\} \sin\eta y\, d\eta,$$

$$E_{xx}(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{[-A_5(x\eta+1)\Omega_1+A_6\Omega_2-A_7l_2^2\eta^2(\Omega_1-\Omega_2)]d_{11}\eta\bar{f}(\eta) + [(A_9-A_8x\eta)\Omega_1-(1-e_3e_9)l_1^{-2}\eta^{-2}\Omega_2+B_1\Omega_2+B_2e_6\Omega_4 - B_3e_5l_2^2\eta^2(\Omega_1-\Omega_2)-B_4l_2^2\eta^2(\Omega_1-\Omega_4) - B_3l_2^2\eta^2(\Omega_2-\Omega_4)]b_{11}\eta\bar{g}(\eta)\Big\} \cos\eta y\, d\eta+b^0,$$

$$E_{yy}(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \{[A_5(x\eta-1)\Omega_1 + B_5\Omega_2 + A_7 l_2^2\eta^2(\Omega_1-\Omega_2)]d_{11}\eta\bar{f}(\eta)$$
$$+[(A_9+A_8x\eta)\Omega_1 - e_{10}l_1^{-2}\eta^{-2}\Omega_2 + B_6\Omega_2 - B_2e_6\Omega_4 + B_3e_5l_2^2\eta^2(\Omega_1-\Omega_2)$$
$$+B_4l_2^2\eta^2(\Omega_1-\Omega_4)+B_3l_2^2\eta^2(\Omega_2-\Omega_4)]b_{11}\eta\bar{g}(\eta)\}\cos\eta y\, d\eta + b^0,$$

$$E_{xy}(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \{[A_5x\eta\Omega_1 + A_7l_2^2\eta^2(\Omega_1-\Omega_3)]d_{11}\eta\bar{f}(\eta) + A_8x\eta\Omega_1$$
$$+(B_2+B_4l_2^2\eta^2)(\Omega_1-\Omega_5)+(B_7+B_3e_5l_2^2\eta^2)(\Omega_1-\Omega_3)$$
$$+B_3l_2^2\eta^2(\Omega_3-\Omega_5)]b_{11}\eta\bar{g}(\eta)\}\sin\eta y\, d\eta,$$

$$E_{yx}(x,y) = E_{xy}(x,y) + \sqrt{\frac{2}{\pi}}\int_0^\infty B_8\Omega_5 b_{11}\eta\bar{g}(\eta)\sin\eta y\, d\eta.$$

The coefficients, $A_i$, $B_i$ and $e_i$ appearing in (3.10) are all made of dimensionless ratios of the material constants. They are defined as follows, $e_1, e_2, \ldots, e_6$ are defined in (3.9),

$$\begin{aligned}
e_7 &= \frac{b_{44}}{b_{11}}, & e_8 &= \frac{d_{44}^2}{b_{11}c_{44}},\\
e_9 &= \frac{d_{11}d_{44}}{b_{11}c_{44}}, & e_{10} &= \frac{b_{12}}{b_{11}} - \frac{d_{12}d_{11}}{b_{11}c_{11}},\\
A_1 &= 2e_1(1-e_1), & A_2 &= 2(e_1-e_2),\\
A_3 &= -2(e_1-e_2)(1+e_4l_1^{-2}l_2^2), & A_4 &= -2(e_1^2+e_2),\\
A_5 &= 2e_2(1-e_1), & A_6 &= e_4(1-e_3^{-1}e_9^{-1})l_1^{-2}l_2^2, \qquad (3.11)\\
A_7 &= 2e_4\left(e_2 - \frac{b_{44}c_{11}}{d_{11}^2}\right)l_1^{-2}l_2^2, & A_8 &= 2(1-e_1)e_8,\\
A_9 &= -2(e_1-e_2)e_9 - 2(1-e_2)e_1e_9, & B_1 &= (1-e_3e_9)(e_5-e_6)(e_7-e_8)l_1^{-2}l_2^2,\\
B_2 &= 2(e_7-e_8), & B_3 &= 2(e_7-e_1e_9),\\
B_4 &= 2(e_1-e_2)e_6e_9, & B_5 &= e_4\left(\frac{d_{12}}{d_{44}} - \frac{b_{12}c_{14}}{d_{11}^2}\right)l_1^{-2}l_2^2,\\
B_6 &= (e_5-e_6)e_{10}l_1^{-2}l_2^2 + 2(e_7-e_1e_9), & B_7 &= -2\left(e_7 - \frac{d_{11}d_{44}}{b_{44}c_{11}}\right),\\
B_8 &= 2e_6\frac{b_{77}}{b_{11}}.
\end{aligned}$$

3.3 *System of dual integral equations*

By comparing the boundary conditions of the auxiliary problem, equations (3.5) with those of the mixed boundary value problem for the crack, equations (3.4), one sees that the first three conditions are common to both problems. Hence, regarding $f(y)$, $g(y)$ or equivalently $\bar{f}(\eta)$, $\bar{g}(\eta)$ as the basic unknowns, two dual integral equations result from the requirement that the general solution of the auxiliary problem, equations (3.10) satisfy the remaining two conditions in equations (3.4). Introducing the normalization with respect to the half crack length $L$ as:

$$x^* = x/L, \qquad y^* = y/L, \qquad l_i^* = l_i/L, \qquad \eta^* = \eta L, \tag{3.12}$$

and thereafter dropping the stars, the system of dual integral equations reads: for $0 \leq |y| < 1$

$$\begin{aligned}
&\int_0^\infty \{[A_1 + A_2 e_4 l_2^2 \eta^2 (1-\eta(\eta^2+l_1^{-2})^{-\frac{1}{2}})] c_{11} \eta \bar{f}(\eta) \\
&\quad + [A_1 - A_2 \eta(\eta^2+l_1^{-2})^{-\frac{1}{2}} - A_2(e_5-e_6) l_2^2 \eta^2 (1-\eta(\eta^2+l_1^{-2})^{-\frac{1}{2}})] d_{11} \eta \bar{g}(\eta)\} \cos \eta y \, d\eta = 0, \\
&\sqrt{\frac{2}{\pi}} \int_0^\infty \{[-A_5 + A_6 \eta(\eta^2+l_1^{-2})^{-\frac{1}{2}} - A_7 l_2^2 \eta^2 (1-\eta(\eta^2+l_1^{-2})^{-\frac{1}{2}})] d_{11} \eta \bar{f}(\eta) \\
&\quad + [(e_3 e_9 - 1)\eta^{-1}(l_1^2\eta^2+1)^{-\frac{1}{2}} + A_9 + B_1 \eta(\eta^2+l_1^{-2})^{-\frac{1}{2}} + B_2 e_6 \eta(\eta^2+l_2^{-2})^{-\frac{1}{2}} \\
&\quad - B_3 e_5 l_2^2 \eta^2 (1-\eta(\eta^2+l_1^{-2})^{-\frac{1}{2}} - B_4 l_2^2 \eta^2 (1-\eta(\eta^2+l_1^2)^{-\frac{1}{2}} - B_3 l_2^2 \eta^2 (\eta(\eta^2+l_1^{-2})^{-\frac{1}{2}} \\
&\quad - \eta(\eta^2+l_2^{-2})^{-\frac{1}{2}})] b_{11} \eta \bar{g}(\eta)\} \cos \eta y \, d\eta = -b^0 L,
\end{aligned} \tag{3.13a}$$

and for $1 \leq |y|$

$$\int_0^\infty \bar{f}(\eta) \cos \eta y \, d\eta = 0, \qquad \int_0^\infty \bar{g}(\eta) \cos \eta y \, d\eta = 0. \tag{3.13b}$$

It is seen that equations (3.13) consist of two pairs of dual integral equations with trigonometric kernels. Erdogan and Bahar [5] presented a method of solution of systems of dual integral equations as a generalization of the procedure due to Tranter [15] for a single pair. This method results in the reduction of a system of simultaneous dual integral equations to an infinite set of algebraic equations.

Consider the integrals [16]:

$$\int_0^\infty \eta^{-1} J_{2k-1}(\eta) \cos \eta y \, d\eta = \begin{cases} (2k-1)^{-1} \cos[(2k-1)\sin^{-1}(y)], & y \leq 1 \\ 0 & y > 1 \end{cases} \tag{3.14}$$

$$\frac{2}{(2h-1)\pi} \int_0^1 \cos[(2h-1)\sin^{-1}(y)] \cos \eta y \, d\eta = \eta^{-1} J_{2h-1}(\eta),$$

where $J_{2k-1}(\eta)$ is the Bessel function of the first kind and $h$, $k$ are positive integers.

By expressing

$$\bar{f}(\eta) = \eta^{-1} \sum_{k=1}^{\infty} M_k J_{2k-1}(\eta), \qquad \bar{g}(\eta) = \eta^{-1} \sum_{k=1}^{\infty} N_k J_{2k-1}(\eta), \tag{3.15}$$

the pair of equations (3.13b) are identically satisfied in view of the property given by the first equation of (3.14). The remaining pair of equations (3.13a), after multiplying by $[2/(2h-1)\pi] \cos[(2h-1)\sin^{-1}(y)]$ and integrating with respect to $y$ in the interval (0–1), reduce to the following system of infinite algebraic equations:

$$\sum_{k=1}^{\infty} \{(-A_1 I_{1,hk} - A_2 e_4 l_1^2 I_{2,hk}) c_{11} M_k + [-A_1 I_{1,hk} + A_2 l_1 I_{2,hk} + A_2(e_5 - e_6) l_1^2 I_{4,hk}] d_{11} N_k\} = 0,$$

$$\sum_{k=1}^{\infty} \{(-A_5 l_1 I_{1,hk} + A_6 l_1^2 I_{2,hk} - A_7 l_1^3 I_{4,hk}) d_{11} M_k + [(e_3 e_9 - 1) I_{0,hk} + A_9 l_1 I_{1,hk} \tag{3.16}$$

$$+ B_1 l_1^2 I_{2,hk} + B_2 e_6 l_1^2 I_{3,hk} - B_3 e_5 l_1^3 I_{4,hk} - B_4 l_1^3 I_{5,hk} - B_3 l_1^4 I_{6,hk}] b_{11} N_k\} = -b^0 l_1 \delta_{h1}/2,$$

where

$$I_{0,hk} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} (l_1^2\eta^2 + 1)^{-\frac{1}{2}} \eta^{-2} J_{2h-1}(\eta) J_{2k-1}(\eta)\, d\eta,$$

$$I_{1,hk} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \eta^{-1} J_{2h-1}(\eta) J_{2k-1}(\eta)\, d\eta,$$

$$I_{2,hk} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} (l_1^2\eta^2 + 1)^{-\frac{1}{2}} J_{2h-1}(\eta) J_{2k-1}(\eta)\, d\eta,$$

$$I_{3,hk} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{l_2}{l_1} \int_0^{\infty} (l_2^2\eta^2 + 1)^{-\frac{1}{2}} J_{2h-1}(\eta) J_{2k-1}(\eta)\, d\eta,$$

$$I_{4,hk} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{l_2}{l_1}\right)^2 \int_0^{\infty} \eta \left(1 - \frac{l_1 \eta}{(l_1^2\eta^2 + 1)^{\frac{1}{2}}}\right) J_{2h-1}(\eta) J_{2k-1}(\eta)\, d\eta, \tag{3.17}$$

$$I_{5,hk} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{l_2}{l_1}\right)^2 \int_0^{\infty} \eta \left(1 - \frac{l_2 \eta}{(l_2^2\eta^2 + 1)^{\frac{1}{2}}}\right) J_{2h-1}(\eta) J_{2k-1}(\eta)\, d\eta,$$

$$I_{6,hk} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{l_2^2}{l_1^3} \int_0^{\infty} \eta \left(\frac{l_1 \eta}{(l_1^2\eta^2 + 1)^{\frac{1}{2}}} - \frac{l_2 \eta}{(l_2^2\eta^2 + 1)^{\frac{1}{2}}}\right) J_{2h-1}(\eta) J_{2k-1}(\eta)\, d\eta.$$

In obtaining the right hand side of the second equation in (3.16), use is made of the following identity

$$\frac{2}{(2h-1)\pi} \int_0^1 \cos[(2h-1)\sin^{-1}(y)]\, dy = \begin{cases} \frac{1}{2} & h = 1 \\ 0 & h \neq 1 \end{cases} \tag{3.18}$$

At this point, the boundedness of the integrals defined in (3.17) must be considered. It is known that the Weber–Schafheitlin integral [17]

$$\int_0^{\infty} \eta^{\sigma} J_{\mu}(\eta) J_{\nu}(\eta)\, d\eta$$

is bounded for $\sigma \leq 0$. Therefore the first four integrals in (3.17) are bounded. The integrals $I_{4,hk}$, $I_{5,hk}$ and $I_{6,hk}$ appear to involve integrands of the type $\eta J_\mu(\eta)J_\nu(\eta)$. However by noting that

$$\eta\left(1-\frac{l\eta}{(l^2\eta^2+1)^{\frac{1}{2}}}\right) = \frac{\eta}{(l^2\eta^2+1)^{\frac{1}{2}}(l\eta+1)}\left(1-\frac{1}{2}\frac{l^2\eta^2}{(l\eta+1)^2}\cdots\right),$$

$$\frac{\eta}{l_1}\left(\frac{l_1\eta}{(l_1^2\eta^2+1)^{\frac{1}{2}}}-\frac{l_2\eta}{(l_2^2\eta^2+1)^{\frac{1}{2}}}\right) = \frac{\eta^2}{(l_1^2\eta^2+1)^{\frac{1}{2}}(l_2^2\eta^2+1)^{\frac{1}{2}}} \tag{3.19}$$

$$\times\left[\frac{1}{l_2\eta+1}\left(1-\frac{1}{2}\frac{l_2^2\eta^2}{(l_2\eta+1)^2}\cdots\right)-\frac{l_2/l_1}{(l_1\eta+1)}\left(1-\frac{1}{2}\frac{l_1^2\eta^2}{(l_1\eta+1)^2}\cdots\right)\right],$$

one sees that these indefinite integrals are convergent, since the integrands behave as $\eta^{-1}J_\mu(\eta)J_\nu(\eta)$.

By substituting coefficients $M_k$, $N_k$, which must be determined from the system of algebraic equations (3.16), into (3.15) for $\bar{f}(\eta)$ and $\bar{g}(\eta)$, then substituting the latter into (3.10), one obtains the fields satisfying the mixed boundary value problem (3.4) for the free linear crack.

However, the chief interest of the investigation in this section is to study the behaviour of the fields along the axis of the crack ($y$ axis), in particular, the behaviour of the functions near the tip of the crack. A detailed examination of (3.10), with consideration of $\bar{f}$, $\bar{g}$ given by (3.15) and relations (3.19), reveals that $\mathbf{u}$, $\mathbf{P}$ and $\phi$ are convergent along the axis ($x = 0$), since the behaviour of the dominant term for large $\eta$ in the integrals involved is of the following form [17]

$$\int_0^\infty \eta^{-1}J_{2k-1}(\eta)\begin{matrix}\sin y\eta\\ \cos y\eta\end{matrix}\,d\eta = (2k-1)^{-1}\begin{matrix}\sin\\ \cos\end{matrix}[(2k-1)\sin^{-1}(y)], \qquad y<1,$$
$$= \frac{\begin{matrix}\sin\\ \cos\end{matrix}[(2k-1)\pi/2]}{(2k-1)[y+\sqrt{(y^2-1)}]^{2k-1}}, \qquad y\geq 1. \tag{3.20}$$

However, the dominant term for large $\eta$ in the integral expressions of $t_{xx}$, $t_{yy}$, $E_{xx}$, $E_{yy}$ and $E_{yx}$ along the $y$ axis behaves as the integral [17]

$$\int_0^\infty J_{2k-1}(\eta)\begin{matrix}\sin y\eta\\ \cos y\eta\end{matrix}\,d\eta = \frac{\begin{matrix}\sin\\ \cos\end{matrix}[(2k-1)\sin^{-1}(y)]}{\sqrt{(1-y^2)}}, \qquad y<1,$$
$$= \frac{\begin{matrix}+\cos\\ -\sin\end{matrix}[(2k-1)\pi/2]}{\sqrt{(y^2-1)}\,.\,[y+\sqrt{(y^2-1)}]^{2k-1}}, \qquad y\geq 1. \tag{3.21}$$

Thus it is seen that along the $y$ axis, $t_{xx}$, $t_{yy}$, $E_{xx}$, $E_{yy}$ and $E_{yx}$ are divergent near the crack tip and behave as $0(\varepsilon^{-\frac{1}{2}})$ as $y = 1+\varepsilon$, $\varepsilon \to 0$. The order of the singularity is similar to those for the stresses at the crack tip in classical elasticity theory [18] and in couple stress theory [6]. Nevertheless, the surface energy density $T$, equation (1.4), is still bounded on the surface of the crack.

In conclusion it should be observed that the surface effects are confined to the immediate neighbourhood of the boundaries and decay exponentially as $e^{-x/l_1}$ where $l_1$ is a length parameter of the order of magnitude of interatomic distances.

For an interior surface, the surface energy density is reduced by an amount proportional to $l_1/R$ where $R$ is the radius of curvature, however, this change is very small in the region where the continuum hypothesis is valid. Furthermore, the "pressure" on a curved surface has the same form as the one given by the classical Laplace formula [12].

In the case of a free linear crack, the presence of the surface energy alone induces a stress singularity of the type $\varepsilon^{-\frac{1}{2}}$ at the crack tip as $\varepsilon \to 0$, nevertheless the surface energy density remains bounded. This singularity is of the same type as that obtained for crack problems in classical elasticity and couple stress theory, however in the latter cases the stresses are induced by the external forces.

In the present theory the presence of the polarization gradient term in the internal energy density function leads naturally to the concept of a surface energy density, which in turn allows one to investigate the effects of surface curvature and discontinuity on the induced fields.

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**Абстракт**—Для расчета двух типов задач касающихся цилиндрических и сферических трещин и свободной линейной щели в плоском деформированном состоянии, используется теория Миндлина упругих диэлектриков и значения коэффициентов материала, получены Аскаром, Ли и Какмаком. Для случая двух трещин, плотность поверхностной энергии деформации и поляризации определяется так, что она изменяется просто пропорционально к параметру длины $l_1$ пррядка величины междуатомного рассмояния и обратно пропорционально к радиусу кривизны трещины. Так, как для щели получается сингулярность порядка $(\epsilon^{-\frac{1}{2}}$ при $(\epsilon \to 0$ и при отсутствии всех внешних сил. Эта сингулярность оказывается такого же самого рода, как сингулярность, вытекающая из теории упругости и теории моментных напряжений. Но однако плотность поверхностной энергии ограничена на поверхности щели.